TRIANGULAR NUMBERS & COMBINATIONS

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Triangular numbers are the count of objects that can be arranged in the form of an equilateral triangle. (Just like how square numbers are the count of objects that can arranged in the form of a square).

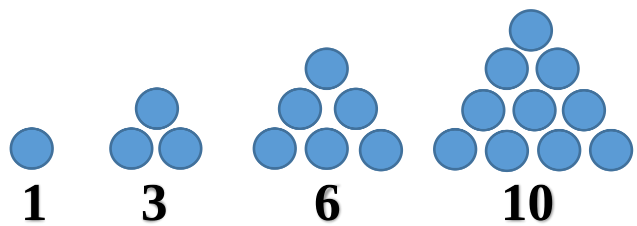

Figure 1: Triangular Numbers

So the triangular number series goes like this – 0, 1, 3, 6, 10, 15, 21, 28, 36, 45 and 55, and so on.

Index	Triangular Number
0th	0
1st	1
2nd	3
3rd	6
4th	10
5th	15
6th	21
…	…

Figure 2: Table of Triangular Numbers

This series looks familiar? First, let us first look at some of the properties of Triangular Numbers.

Property #1:

Let us find the difference between consecutive triangular numbers.

0 1 3 6 10 15

1 - 0 3 – 1 6 – 3 10 – 6 15 - 10

1 2 3 4 5

Figure 3: Difference between consecutive Triangular Numbers

If you observe, the difference between consecutive triangular numbers results in a natural number series 1, 2, 3, 4, 5, 6, and so on. That is, the **n**th triangular number is obtained by adding "**n**" to the **(n-1)**th triangular number.

Ex: 5th Triangular Number = 5 + 4th Triangular Number = 5 + 10 = 15.

In other words, the **n**th triangular number is the sum of all numbers from **1** to **n**.

Ex: 4th Triangular Number = Sum of all numbers from 1 to 4 = 1 + 2 + 3 + 4 = 10.

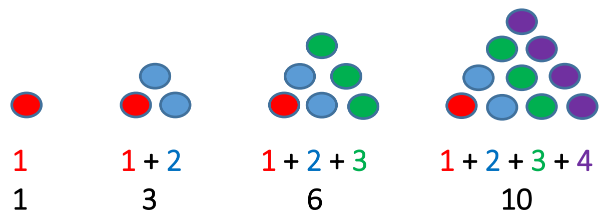

Figure 4: Summation of numbers from 1 to n

Thus the series of triangular numbers is nothing but the series of summation of numbers from 1 to n. Did you spot this when you looked at Figure 2?

Property #2:

Let us now find the sum of consecutive triangular numbers.

0 1 3 6 10 15

0 + 1 1 + 3 3 + 6 6 + 10 10 + 15

1 4 9 16 25

Figure 5: Sum of consecutive Triangular Numbers

If you observe, the sum of consecutive triangular numbers results in a series of square numbers 1, 4, 9, 16, 25, 36, and so on. That is, the sum of the **n**th triangular number and the **(n+1)**th triangular number is nothing but the **(n+1)**th square number.

Ex: 3rd Triangular Number + 4th Triangular Number = 6 + 10 = 16 = 4^2 = 4th Square Number.

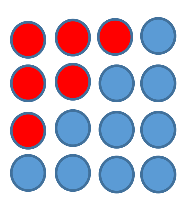

Figure 6: Two consecutive Triangular Numbers make a Square Number.

Let us now look at some practical applications of Triangular Numbers.

Handshake Puzzle – Consider the well-known handshake puzzle. If there are 5 folks who meet up for a party, and if each one of them shakes hand with everyone else, then how many handshakes happen in total?

There are different ways of solving this puzzle. Let us look at one simple way.

Out of the 5 folks, one of them shakes hand with the other 4 folks, and goes for a coffee drink. So 4 handshakes happen. Let us represent each handshake as a circle.

Figure 7: 4 Handshakes in total

Now, out of the 4 folks, one of them shakes hand with the other 3 folks, and goes for a coffee drink. So 3 handshakes happen. Again, represent each handshake as a circle and add it to the previous set of circles.

Figure 8: 4 + 3 Handshakes in total

Now, out of the 3 folks, one of them shakes hand with the remaining 2 folks, and goes for a coffee drink. So 2 handshakes happen. Again, represent each handshake as a circle and add it to the previous set of circles.

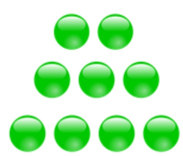

Figure 9: 4 + 3 + 2 Handshakes in total

Do you see the pattern building up?

Now the remaining 2 folks shake hands with each other, and go together for a coffee drink. So 1 handshake happens. And we get the final set of circles as below.

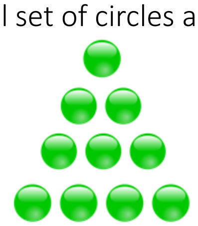

Figure 10: 4 + 3 + 2 + 1 Handshakes in total

What have we got here? A Triangular Number! Yes, the 4^{th} Triangular Number = 10. This represents the number of handshakes that happen in a party of 5 folks where each one shakes hand with all others. In general, in a party of "**n**" folks, the total number of handshakes would be the **(n-1)**th Triangular Number.

Let us take one more puzzle.

Full Mesh Network Puzzle – Given 5 cities as shown below, how many roads need to be constructed so that there is a direct road connectivity from any city to any other city.

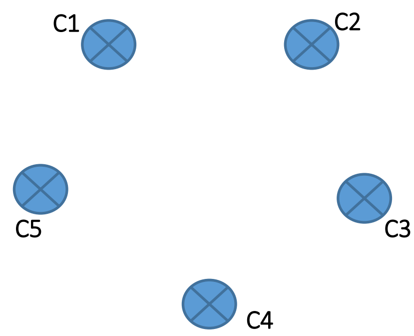

Figure 11: 5 Cities without any road connectivity.

Again, there are different ways of solving this puzzle. Let us look at one simple way.

Let's construct roads from city C1 to all other cities. So we would need to construct **four** roads.

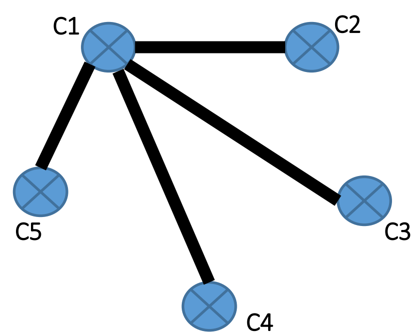

Figure 12: Four new roads connecting C1 to all other cities.

Now let's construct roads from city C2 to all other cities. Since the road from C2 to C1 is already constructed, we need to construct just **three** roads as shown below.

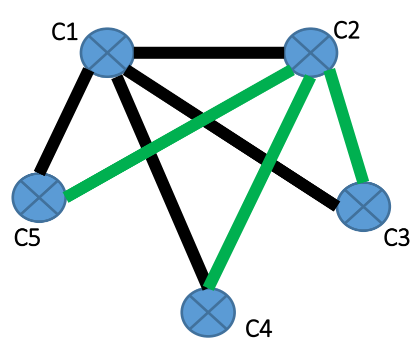

Figure 13: Three new roads ensure that C2 is connected to all other cities.

Continuing in a similar fashion, we can easily see that we need to construct **two** roads from C3 and **one** road from C4 as shown below.

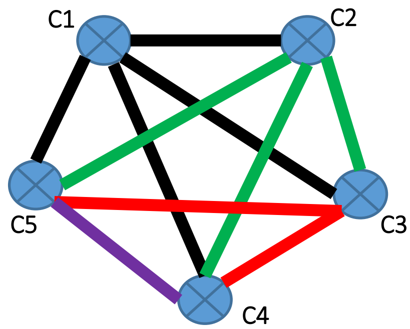

Figure 14: Full Mesh Road Network of 5 cities.

If you observe, the same pattern repeats here. We have 4 + 3 + 2 + 1 = 10 roads that need to be constructed in total. And this is nothing but the 4^{th} Triangular Number! In general, if there are "**n**" cities, then the number of roads that need to be constructed would be the **(n-1)**th Triangular Number.

So we see that Triangular Numbers come up surprisingly in many of these different puzzles. However, if we model these puzzles mathematically, they all belong to one common category of puzzles, which we can solve using a simpler technique. Let us find that out.

From **Property#1**, we know that the **x**th triangular number is nothing but the sum of all numbers from **1** to **x**. The sum of numbers from 1 to "x" is given by the formula

S = x * (x + 1) / 2

Let us multiply the numerator and the denominator by (x -1) factorial. We get

S = x * (x + 1) * (x – 1)! / (2 * (x – 1)!)

Since x * (x + 1) * (x – 1)! = (x + 1)! , we get

S = (x + 1)! / (2 * (x – 1)!)

Let n = (x + 1). Therefore (x – 1) = (n – 2). Replacing with n, we get

S = n! / (2 * (n – 2)!)

Since 2 can also be written as 2! , we get

S = n! / (2! * (n – 2)!)

Does this formula sound familiar? Yes, "Combinations". That's correct.

The formula for the number of combinations of "r" objects out of a total "n" objects is

$^{n}C_{r}$ **= n! / (r! * (n – r)!)**

And if we replace “r” with “2” in the above formula, we get the same formula as we got for “**S**”.

In other words, Triangular Number series is nothing but a series of combinations of the form ${}^{n}C_{2}$.

Let us tabulate the values of ${}^{n}C_{2}$ for different values of n.

n	${}^{n}C_{2}$	Value	Triangular Number Index
2	2! / (2 * 0!)	1	1^{st}
3	3! / (2 * 1!)	3	2^{nd}
4	4! / (2 * 2!)	6	3^{rd}
5	5! / (2 * 3!)	10	4^{th}
6	6! / (2 * 4!)	15	5^{th}

Figure 15: Table depicting Triangular Numbers and Combinations.

If you observe the X^{th} Triangular Number is given by the formula ${}^{(X+1)}C_{2}$

It’s no wonder that we let **n = (x + 1)** while deriving the formula above. Right?

Well, it’s time to put the things, we have learnt so far, into action.

Going back to the Handshake Puzzle, we can solve that puzzle using combinations. The answer to the puzzle is the total number of different ways of selecting two people out of 5. This is nothing but ${}^{5}C_{2} = 10$.

Similarly, for the Full Mesh Network puzzle, the answer to the puzzle is the total number of different ways of selecting two cities out of 5, and again this is nothing but ${}^{5}C_{2} = 10$.

Even if one doesn’t know about Permutations and Combinations, these kind of puzzles can still be solved using the Triangular Number approach, once the pattern is recognized.

Coming to the last piece in this article, I have an interesting puzzle for you. Consider a long strip of paper as shown below.

Figure 16: Number Strip Puzzle.

How many different number strips can you make out of this, if you are allowed to only cut the strip, but not join the smaller cut strips? For example, you can make “12” or “2345” or “45678” and so on. However “13”, for example, is not allowed since you have to cut two smaller strips “1” and “3”, and join them to make 13.

Can you solve this puzzle using the concept of Triangular Numbers, as well as using the concept of Combinations? You need to model the puzzle differently if you want to use combinations, but ultimately you will get the same answer in both the approaches.